Symbolic Computation in the Analysis of the Dynamics of an Orbital Gyrostat

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Abstract

The paper presents the possibilities of symbolic computation in the study of the dynamics of rotational motion along a circular orbit of a gyrostat in a Newtonian central field of forces. In accord with the problem of Lyapunov's stability from the equations of perturbed motion in the first approximation, in the space of the introduced parameters, the regions are found in which there is gyroscopic stabilization of unstable equilibria. The results were obtained with the help of applied software and functions of symbolic-numerical modeling of the computer algebra system *Mathematica*.

Keywords

Gyrostat, symbolic-numerical modeling, stability of equilibria, gyroscopic stabilization, software package of computer algebra.

1. Introduction

The process of solving problems on a computer in an analytical form is called symbolic computation. Symbolic computation [1] have long passed into the category of computer methods with a variety of applications (see, for example, [2]).

The classical problem of the influence of the structure of forces on the stability of the equilibria of mechanical systems [3] began to develop in the 19th century – the effect of gyroscopic stabilization was discovered. Nevertheless, the problem remains relevant: see, for example, the review in [4] among the large number of publications on this topic. With the help of symbolic computations, the present paper investigates the dynamics of an orbital gyrostat (with an arbitrary inertia ellipsoid) and the question of the possibility of gyroscopic stabilization of its unstable equilibrium positions.

The rigid body with the fixed axis of a statically and dynamically balanced flywheel rotating about that axis with a constant relative angular velocity is a gyrostat. The system moves along the circular Keplerian orbit in a central Newtonian field of forces around the gravitational center.

The stabilization of equilibria using the equations of the first approximation, that is presented here, continues and supplements the studies performed earlier for oblate and prolate axisymmetric gyrostat (see, for example, [5]).

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2. Construction of a Symbolic Model and Stability Conditions

By the construction of a symbolical model, one implies the obtaining of nonlinear and linearized differential equations of motion in analytical form in computer memory. The software package (SP) [6] used in this paper is designed for modeling and qualitative analysis in symbolic form of dynamic systems (in particular, the systems of interconnected absolutely rigid bodies). This applied software, the functional description and application technology of which is given in [7, 8], is a set of interactive programs executed in the interpretation mode in the computer algebra system (CAS) *Mathematica* environment.

Consider the position of equilibrium ($\dot{\alpha} = 0$, $\dot{\beta} = 0$, $\dot{\gamma} = 0$) in regard to orbital coordinate system (OCS) in general form:

$$\alpha = \alpha_0 = \text{const}\,, \quad \beta = \beta_0 = \text{const}\,, \quad \gamma = \gamma_0 = \text{const}\,. \tag{1}$$

Here $\alpha\,, \beta\,, \gamma$ are the aircraft angles that specify the orientation of the principal central axes relative to the OCS.

The equations of motion of the satellite-gyrostat with respect to its center of mass in Euler form are quite well-known (see, for example, [9]). With the help of SP [6], the following results in an analytical form in computer memory are obtained:

(a) kinetic energy and force function of the approximate Newtonian field of gravitation;

(b) nonlinear differential equations in Lagrange form of the second kind describing the motion of an orbital gyrostat;

(c) existence conditions of equilibrium (1).

For example, Figure 1 shows the least bulky equation of motion with respect to the generalized coordinate γ (i.e. this is one of the three equations (b)).

$$\begin{aligned}
&A\,(\ddot{\gamma} + \ddot{\alpha}\,\mathrm{Sin}[\beta]) + \\
&\quad \frac{1}{2}\Big(2\,\mathrm{Cos}[\beta]\,(A + (-B + C)\,\mathrm{Cos}[2\,\gamma])\,\dot{\alpha}\,\dot{\beta} + (B - C)\,\mathrm{Cos}[\beta]^2\,\dot{\alpha}^2\,\mathrm{Sin}[2\,\gamma] + (-B + C)\,\dot{\beta}^2\,\mathrm{Sin}[2\,\gamma]\Big) + \\
&\quad \omega\,\Big(\mathrm{Cos}[\beta]\,\dot{\alpha}\,((B - C)\,\mathrm{Cos}[\beta]\,\mathrm{Sin}[2\,\gamma] + \mathrm{Sin}[\gamma]\,h_2 + \mathrm{Cos}[\gamma]\,h_3) + \\
&\qquad \dot{\beta}\,(\mathrm{Cos}[\beta]\,(A + (-B + C)\,\mathrm{Cos}[2\,\gamma]) - \mathrm{Cos}[\gamma]\,h_2 + \mathrm{Sin}[\gamma]\,h_3)\Big) + \\
&\quad \omega^2\,\Big((B - C)\,\Big(3\,\mathrm{Cos}[\alpha]\,\mathrm{Cos}[2\,\gamma]\,\mathrm{Sin}[\alpha]\,\mathrm{Sin}[\beta] + \mathrm{Cos}[\gamma]\,\Big(3\,\mathrm{Cos}[\alpha]^2 + \mathrm{Cos}[\beta]^2 - 3\,\mathrm{Sin}[\alpha]^2\,\mathrm{Sin}[\beta]^2\Big)\,\mathrm{Sin}[\gamma]\Big) + \\
&\qquad \mathrm{Cos}[\beta]\,\mathrm{Sin}[\gamma]\,h_2 + \mathrm{Cos}[\beta]\,\mathrm{Cos}[\gamma]\,h_3\Big) = 0
\end{aligned}$$

Figure 1: The equation of motion in Lagrange form with respect to the coordinate γ.

where $A\,, B$, and C are the moments of inertia of the gyrostat relative to its principal central axes, and h_j $(j = \overline{1, 3})$ are the projections of gyrostatic moment vector of system onto these axes. See [10] for a detailed description of the system's motion and an explicit form of the conditions (c).

Necessary conditions of stability for the equilibrium can be obtained from the equations of perturbed motion in the first approximation. The linearized equations of perturbed motion in vicinity of (1) look this way:

$$M\,\ddot{q} + G\,\dot{q} + K\,q = 0\,, \tag{2}$$

where $q = \left(\overline{\alpha},\, \overline{\beta},\, \overline{\gamma}\right)^T$ is the column vector of deviations of generalized coordinates from the unperturbed motion (1); M is a positive definite symmetric matrix of kinetic energy; G is a skew-symmetric matrix of gyroscopic forces; and K is a symmetric matrix of potential forces.

The structure and the explicit form of the elements of the matrices $M,\ G,\ K$ are presented in [10]. All derivatives in (2) are calculated by dimensionless time $\tau = \omega t$. Here ω is the module of orbital angular velocity.

The characteristic equation: $\det\left(M\lambda^2 + G\lambda + K\right) = v_3\lambda^6 + v_2\lambda^4 + v_1\lambda^2 + v_0 = 0$ of system (2) contains λ only in even degrees. The stability of equilibrium (1) takes place when all roots with respect to λ^2, being simple, will be real negative numbers. The algebraic conditions providing specified properties of roots (*necessary conditions of stability*), represent the system of inequalities:

$$\begin{cases} v_3 \equiv \det M > 0\,, \quad v_2 > 0\,, \quad v_1 > 0\,, \quad v_0 \equiv \det K > 0\,, \\ Dis \equiv v_2^2 v_1^2 - 4v_1^3 v_3 - 4v_2^3 v_0 + 18 v_3 v_2 v_1 v_0 - 27 v_0^2 v_3^2 > 0\,. \end{cases} \tag{3}$$

Emphasize that the construction of the symbolic linearized model (2) (i.e., obtaining in the analytical form the elements of the matrices M, G, K), the calculation of the coefficients $v_i\ (i = \overline{0,3})$ and the discriminant Dis from (3) was also performed using the SP [6].

3. Parametric Analysis of Gyroscopic Stabilization Conditions

In [11], using the algorithms of constructing Gröbner bases, all the equilibrium positions in regard to OCS of a satellite-gyrostat are determined analytically or numerically for three special cases. For these cases, the gyrostatic moment vector is in one of the planes formed by the satellite's principal central axes of inertia. For example, in case ($h_3 = 0\,,\ h_1 \neq 0\,,\ h_2 \neq 0$), there is the equilibrium orientation:

$$\begin{cases} \alpha = \alpha_0 = \pi/2\,, \quad \gamma = \gamma_0 = 0\,, \\ \beta = \beta_0 = \text{const} : \quad h_2 \sin\beta_0 - \cos\beta_0\,(\,h_1 + 4\,(A - B)\sin\beta_0) = 0\,. \end{cases} \tag{4}$$

Let us parametrize the problem. Without loss of generality, let $h_i > 0\,,\ (i = 1, 2)$, and $B > A > C$ for definiteness. Let us introduce dimensionless parameters:

$$H_1 \equiv \frac{h_1}{B}\,; \quad H_2 \equiv \frac{h_2}{B}\,; \quad J_A \equiv \frac{A}{B}\,; \quad J_C \equiv \frac{C}{B}\,; \quad p_c \equiv \cos\beta_0\,; \quad p_s \equiv -\sin\beta_0\,. \tag{5}$$

The values of the parameters belong to the intervals:

$$\begin{gathered} H_i > 0\,,\ (i = 1, 2)\,; \qquad 1/2 < J_A < 1\,,\ \ 1 - J_A < J_C < J_A\,; \\ -1 < p_c < 1\,,\ \left(p_c \neq 0\,,\ p_s = \pm\sqrt{1 - p_c^2}\,\right). \end{gathered} \tag{6}$$

According to Kelvin–Chetaev's theorems [3], studying the questions on stability of equilibria begins with an analysis of the matrix of potential forces. For applied problems of spacecraft dynamics, one usually sets the distribution of masses in the system, under which the initial matrix of potential forces will be positive definite. Further, due to the influence of dissipative forces, the asymptotic stability of motion is ensured by the Lyapunov theorem. However, potentially unstable systems may also be of interest, for example, because of the possibility of nonstandard situations in orbit.

The equation from (4) in notations (5) is resolved with respect to the parameter H_1 as follows: $H_1 = p_s\,(4\,(J_A - 1) - H_2/p_c)$. Considering last relation and notations (5), the matrix of potential forces for the equilibrium (4) takes the form:

$$K = \begin{pmatrix} 3\left(J_C - p_s^2 - J_A p_c^2\right) & 0 & -3\,(J_C - 1)\,p_s \\ 0 & K_{22} & 0 \\ -3\,(J_C - 1)\,p_s & 0 & K_{33} \end{pmatrix}, \tag{7}$$

where $K_{22} = H_2/p_c + 4\,(1 - J_A)\,p_c^2$; $K_{33} = H_2 p_c + (1 - J_C)\big(2\left(p_c^2 - p_s^2\right) - 1\big)$.

It is not difficult to show that the principal diagonal first-order minor of the matrix K from (7) on the intervals (6) is negative. Hence, the matrix of potential forces is not positive definite and equilibrium (4) will be unstable.

The parameter p_s enters the coefficients of the system's characteristic equation only in even degrees. Let us eliminate it, considering $p_c^2 + p_s^2 = 1$. Let us write down these coefficients depending on four parameters J_A , J_C , p_c , H_2 in an explicit form:

$$\begin{gathered} v_3 \equiv \det M = J_A J_C p_c^2\,; \qquad v_2 = H_2^2\left(J_A - (J_A - 1)\,p_c^2\right) + \\ + H_2 p_c\left((J_A - 1)\,p_c^2\,(6J_A + J_C - 2) - J_A\,(6J_A + J_C - 7)\right) + \\ + p_c^2\left(-\,(J_A - 1)\,p_c^2\left(2\,(3J_A + 1)\,J_C + (1 - 3J_A)^2 - 3J_C^2\right) + \right. \\ \left. +\,(J_A\,(3J_A - 2) - 3)\,J_C + 9J_A\,(J_A - 1)^2 + 3J_C^2\right); \\ v_1 = H_2^2\left(J_A + 3J_C - 3 - 4\,(J_A - 1)\,p_c^2\right) + H_2 p_c\left(J_A\,(22 - 19J_C) - 3J_A^2 + \right. \\ \left. +\,(J_A - 1)\,p_c^2\,(6J_A + 19J_C - 26) - 3\,(J_C - 7)\,(J_C - 1)\right) + \\ + p_c^2\left((J_A - 1)\,p_c^2\left(6J_A\,(5 - 7J_C) + 9J_A^2 - 3J_C^2 + 34J_C - 31\right) + \right. \\ \left. +\,3\,(J_C - 1)\left((3J_A - 2)\,J_C + 9\,(J_A - 1)^2\right)\right); \\ v_0 \equiv \det K = 3\left(H_2 - 4\,(J_A - 1)\,p_c^3\right) * \\ * \left(H_2\left(J_C - 1 - (J_A - 1)\,p_c^2\right) + p_c\,(J_C - 1)\left(4\,(J_A - 1)\,p_c^2 - 3J_A - J_C + 4\right)\right). \end{gathered} \tag{8}$$

It is known that if the equilibrium position is unstable at potential forces, Kelvin–Chetaev's theorem [3] of influence of gyroscopic forces tells us that gyroscopic stabilization is possible only for systems with an even degree of instability. The evenness (or oddness) of the degree of instability according to Poincare is determined by positivity (or negativity) of the determinant of the matrix of potential forces.

Let us pose the question of the possibility of gyroscopic stabilization of unstable equilibrium (4) under condition $\det K > 0$. With the help of *Mathematica* function:

$$\mathit{Reduce}[\{\,1/2 < J_A < 1\,,\ 1 - J_A < J_C < J_A\,,\ -1 < p_c < 1\,,\ p_c \neq 0\,,\ H_2 > 0\,,\ \det K > 0\,\},\\ \{\,J_A\,,\ J_C\,,\ p_c\,,\ H_2\,\}, \mathrm{Reals}\,]$$

designed to find the symbolic (analytical) solution of the inequalities systems, the region with an even degree of instability is obtained. Due to the solution bulkiness, its presentation is omitted here. An analysis of the solution obtained allows us to conclude the following conclusion.

Proposition 1. *The region with an even degree of instability for equilibrium (4) with the values of parameters* J_A , J_C *from (6) lies in the plane:* $-1 < p_c < 0 \wedge 0 < H_2 < 2$.

Next, we will use two more *Mathematica* functions *RegionPlot*, *RegionPlot3D*, designed, respectively, for a graphical 2D and 3D representations of the solution of the inequalities system.

For the detection of a property of gyroscopic stabilization, it is necessary to find in what part of region with an even degree of instability the remaining inequalities from (3) are fulfilled (except for $v_3 \equiv \det M > 0\,,\ v_0 \equiv \det K > 0$). It is not possible to obtain an analytical solution for the entire system of inequalities (3) because of the large number of parameters and the complexity of the expressions being analyzed. For example, the discriminant Dis (see (3)) of a cubic equation is an 8^{th} degree polynomial in regard to H_2 with the coefficients depending in a complicated manner on the parameters J_A, J_C, and p_c. Their maximum degrees are 12, 8, 18, respectively. Therefore, to simplify the analysis, let us move on to symbolic-numerical modeling for fixed values of one or two parameters.

For certainty, let one of the parameters have the following value: $p_c = -\sqrt{2}/2$. Instead of solving system (3), let us first consider a simpler problem. The region of positivity of coefficients (8) was obtained using the function:

$$\textit{RegionPlot3D}[\,1 - J_A < J_C < J_A \wedge v_0 > 0 \wedge v_1 > 0 \wedge v_2 > 0\,, \{J_A\,, 1/2\,, 3/4\}\,,\\ \{J_C\,,\ 39/100\,,\ 3/4\},\ \{\ H_2\,,\ \sqrt{2}/2\,,\ \sqrt{2}\,\}]$$

and is presented in Figure 2.

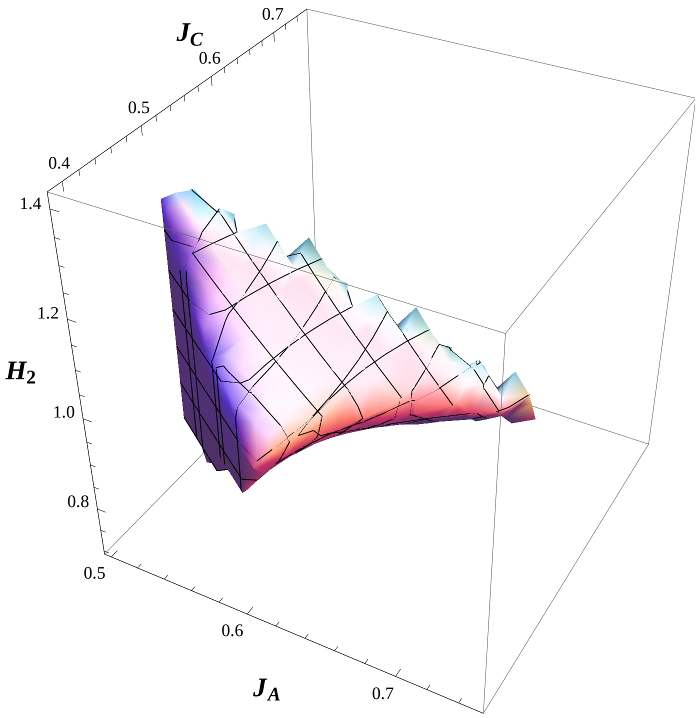

Figure 2: The region of positivity of coefficients (8) for $p_c = -\sqrt{2}/2$.

The boundary values of the parameters J_A, J_C, H_2 were in advance found by the function *Reduce* with a change in their orderliness.

Drawing a conclusion from the three-dimensional symbolic-numerical modeling, we can formulate

Proposition 2. *The gyroscopic stabilization of the unstable equilibrium (4) is possible only for values of the parameters (5) from the intervals:*

$$1/2 < J_A < 3/4 \,\wedge\, 1 - J_A < J_C < J_A \,\wedge -1 < p_c < -1/2 \,\wedge\, 1/2 < H_2 < 2\,.$$

Experience shows that three-dimensional images are often complex and not informative enough to draw conclusions on. Therefore, further parametric analysis of the system of inequalities (3) was carried out with respect to two selected parameters.

In addition to the value of the parameter p_c, let the second parameter have the value: $J_C = 1/2$. Let us construct the regions with an even degree of instability and of gyroscopic stabilization in the parameter plane J_A, H_2 using function

$$\textit{RegionPlot}[\,1/2 < J_A < 1 \wedge 0 < H_2 < 2 \wedge v_0 > 0 \wedge v_1 > 0 \wedge v_2 > 0 \wedge Dis > 0\,,\\ \{J_A\,, 1/2\,,\, 1\}, \{H_2\,,\, 0\,,\, 2\}\,]\,.$$

The result obtained is shown with colour regions in Figure 3.

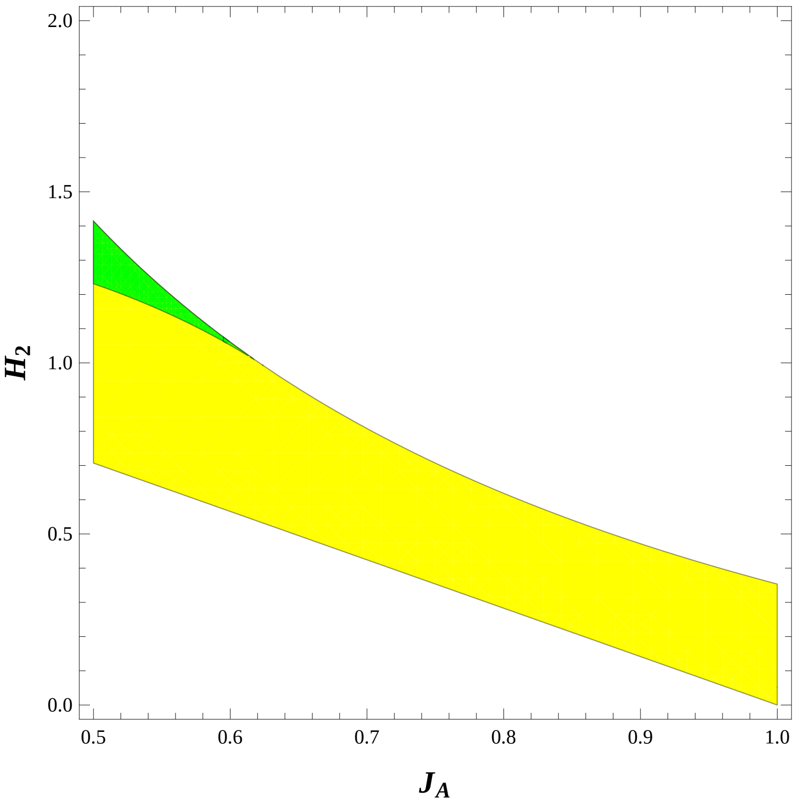

Figure 3: The found areas for $p_c = -\sqrt{2}/2$, $J_C = 1/2$.

The yellow area in the figure is the region with an even degree of instability. Its green part determines the parameter values at which gyroscopic stabilization is possible. Outside the selected regions, the system has an odd degree of instability (i.e. $\det K < 0$) and equilibrium (4) is unstable.

4. Conclusion

By analogy with the parametric analysis presented above, the possibility of gyroscopic stabilization for other equilibrium positions was also examined. It is important to note that this question is not always resolved positively (see, for example, [10]).

The presented results of the dynamics of an orbital gyrostat using symbolic-numerical calculations indicate that the proposed approach expands our capabilities in the study of multi-parameter problems.

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